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ON CERTAIN CASES OF DETERMINATION OF TRAJECTORIES OF PLANE CONSERVATIVE MOTION OF A PARTICLE

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A method is developed for obtaining an incomplete integral of the two-dimensional Hamilton — Jacobi equation for a particle when a particular solution that satisfies a special equation in partial derivatives is known. The equation for the trajectory of dipole particles moving with zero total energy in an arbitrary two-dimensional electrostatic field is obtain, as an example.

The problem of particle motion in a two-dimensional conservative field can be solved in quadratures only in the case of Liouville systems [1]. The range of problems integrable in quadratures can be widened by seeking an incomplete integral of the Hamilton — Jacobi equation containing one free constant and, consequently, defining an isoenergetic set of trajectories. A method is proposed in this paper for the derivation of an incomplete integral using a known particular integral that satisfies some special condition.

Let us conside the motion of a particle of unit mass in a two-dimensional field with potential $\Pi(x, y)$, where x and y are Cartesian coordinates in a plane. The respective Hamilton — Jacobi equation is of the form

$$\tfrac{1}{2}(S_x^2 + S_y^2) + \Pi(x, y) = h \tag{1}$$

where $S(x, y, h, \alpha)$ is the sought complete integral, h is the particle total energy, and α an arbitrary constant.

Let us assume that the particular integral of (1) for some fixed meaning of energy h_0 of form $u(x, y)$ or the integral of form $u(x, y, h)$ has been obtained and that it satisfies the linear homogeneous second order equation in partial derivatives

$$u_{xx} + u_{yy} + au_x + bu_y = 0 \tag{2}$$

In the first case such integral can be found, if

$$\Pi = h_0 - \tfrac{1}{2}(u_x^2 + u_y^2) \tag{3}$$

and $u(x, y)$ satisfies Eq. (2).

The conditions that must be satisfied by functions $a(x, y)$ and $b(x, y)$ will be formulated later.

Let us show how to obtain in this case an integral of (1) of the form $S(x, y, \alpha)$ or $S(x, y, h, \alpha)$, respectively. Since u is a particular integral of (1), hence

$$S_x^2 + S_y^2 = u_x^2 + u_y^2 \tag{4}$$

We seek the integral of Eq. (4) in the form of solution of the following system of linear first order equations:

$$S_x = u_x \sin \varphi + u_y \cos \varphi, \quad S_y = -u_x \cos \varphi + u_y \sin \varphi \tag{5}$$

whose compatibility is satisfied on the strength of (2), and parameter α must appear in function $\varphi(x, y, \alpha)$ which is yet to be determined.

By virtue of system (5) Eq. (4) is satisfied for any function φ. The compatibility condition (5) $S_{xy} = S_{yx}$ leads to the equation

$$u_{xx} + u_{yy} - (\varphi_x \operatorname{tg} \varphi - \varphi_y)u_x - (\varphi_x + \varphi_y \operatorname{tg} \varphi)u_y = 0 \tag{6}$$

The comparison of (6) with (2) yields the following system of two quasilinear equations for φ:

$$\varphi_x \operatorname{tg} \varphi - \varphi_y = -a, \quad \varphi_x + \varphi_y \operatorname{tg} \varphi = -b \tag{7}$$

and the compatibility condition $\varphi_{xy} = \varphi_{yx}$ of system (5) yields the equation

$$(a_y - b_x) \operatorname{tg} \varphi = a^2 + b^2 - a_x - b_y \tag{8}$$

Since the sought integral of Eq. (6) depends on the arbitrary constant α, while functions a and b are independent of α, hence (8) yields a system which imposes on these functions the requirement

$$a_y - b_x = 0, \quad a_x + b_y = a^2 + b^2 \tag{9}$$

By setting

$$a = p_x / p, \quad b = p_y / p \tag{10}$$

we satisfy the first equation of system (9), while the second yields for function p the condition

$$p_{xx} + p_{yy} = 0 \tag{11}$$

When conditions (10) and (11) are satisfied, system (7) becomes completely integrable, which makes it possible to determine its integral of the form $\varphi(x, y, \alpha)$. Thus by seeking φ in the implicit form

$$\Phi(x, y, \varphi) = \text{const} = \alpha \tag{12}$$

we obtain a completely integrable system for functions $\Phi(x, y, \varphi)$ in which φ plays the part of variable

$$\Phi_x \operatorname{tg} \varphi - \Phi_y = a\Phi_\varphi, \quad \Phi_x + \Phi_y \operatorname{tg} \varphi = b\Phi_\varphi \tag{13}$$

By finding some integral of system (13) containing φ, substituting it into (12), and solving the obtained equation for φ, we determine the required integral of (7). Hence, when the particular integral of (1) satisfies Eq. (2) with coefficients defined by (10) and (11), it is possible to find the integral of (1) which in addition to previous variables contains the arbitrary constant α.

As an example of application of this method, we shall consider the plane motion of a particle with the potential energy

$$\Pi = -{}^1/_2\,(u_x^2 + u_y^2) \tag{14}$$

where $u(x, y)$ is an arbitrary harmonic function of the form

$$u_{xx} + u_{yy} = 0 \tag{15}$$

This case represents a wide class of phenomena, such as the motion of electrically neutral atoms and molecules or uncharged metal parts in an electrostatic field with harmonic potential $u(x, y)$.

Substituting the expression for Π in (14) into (1), we obtain

$$(S_x^2 + S_y^2) - (u_x^2 + u_y^2) = 2h \tag{16}$$

When $h = 0$, then $S = u(x, y)$ which satisfies Eq. (15) which is a particular case of (2) when $a = 0$ and $b = 0$ and, consequently $p = \text{const}$ is obviously a particular integral of (16). With these values of a and b $\varphi = \alpha$ can be taken as the integral of system (7). Then the incomplete integral of (16) assumes the form

$$S(x, y, \alpha) = u(x, y)\sin\alpha - v(x, y)\cos\alpha \tag{17}$$

where $v(x, y)$ is a harmonic function $(u_x = v_y,\ u_y = -v_x)$ conjugate of $u(x, y)$.

In this case the incomplete integral is expressed in terms of the electrostatic potential $u(x, y)$ and the stream function $v(x, y)$, while the trajectory of a particle moving in an arbitrary electrostatic field with zero total energy in the plane of field action, can be always be defined in quadratures, since by the Lehman Filet theorem [2] the relation

$$S_\alpha = \beta \tag{18}$$

is the equation of trajectory. Substituting (17) into (18) we obtain the equation of particle trajectory when $h = 0$

$$u(x, y)\cos\alpha + v(x, y)\sin\alpha = \beta$$

For instance, the entry of a particle at arbitrary velocity perpendicular to the field plane of action corresponds to the case of zero energy in that plane.

Let us also consider the motion of dipole particles in axisymmetric electrostatic fields, with zero initial momentum of rotation about the axis.

In cylindrical coordinates r, ψ, z (1) assumes the form

$$(S_r^2 + S_z^2) - (u_r^2 + u_z^2) = 2h \tag{19}$$

with $u(r, z)$ satisfies the axisymmetric Laplace equation

$$u_{rr} + u_{zz} + r^{-1}u_r = 0 \tag{20}$$

Equation (20) satisfies conditions (2), (10), and (11), since in this case $p = r$ (we recall that r and z are now used instead of x and y). When $h = 0$ Eq. (19) has the integral $S = u(r, z)$. In this case system (13) is of the form

$$\Phi_r \operatorname{tg}\varphi - \Phi_z = r^{-1}\Phi_\varphi, \quad \Phi_r + \Phi_z \operatorname{tg}\varphi = 0 \tag{21}$$

Having solving system (21), we obtain

$$\varphi = \operatorname{arctg}\frac{\alpha + z}{r}$$

Hence it is possible to write the integral of system (5) as

$$S = \int_{(r_0, z_0)}^{(r, z)} \frac{(\alpha + z) u_r + r u_z}{\sqrt{r^2 + (\alpha + z)^2}} dr + \frac{(\alpha + z) u_z - r u_r}{\sqrt{r^2 + (\alpha + z)^2}} dz \qquad (22)$$

Substituting now in (18) the expression (22) for S we obtain the trajectories of dipole particles moving with zero energy in the meridional plane of an arbitrary axisymmetric electrostatic field.

In conclusion we shall show how all problems that are integrable by the proposed method can be reduced to the case considered above. This makes it possible to avoid solving systems of differential equations and use the already available results. Let us assume that the integral $u(x, y)$ of Eq. (1) which satisfies Eq. (2) for some function $p(x, y) \neq \text{const}$ has been found when $h = h_0$. We pass to curvilinear coordinates ξ, η

$$w = f(z); \quad w = \xi + i\eta, \quad z = x + iy, \quad f(z) = p(x, y) + iq(x, y)$$

where $q(x, y)$ is a function harmonically conjugate of $p(x, y)$.

When $h = h_0$, Eq. (1) in terms of the new coordinates if of the form

$$(S_\xi^2 + S_\eta^2) - (U_\xi^2 + U_\eta^2) = 0, \quad U(\xi, \eta) = u[x(\xi, \eta), y(\xi, \eta)] \qquad (23)$$

Function $U(\xi, \eta)$ satisfies Eq. (2) expressed in coordinates ξ, η which within the designation of variables is the same as Eq. (20). Equation (23) is the same as (19) when $h = 0$, hence the whole problem reduces to the one already solved. If we extend the axisymmetric case using arbitrary harmonic functions $p(x, y)$, taking into consideration also the case of $p(x, y) = \text{const}$, we obtain all problems that are integrable by the expounded method.

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